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Asymptotic solutions of time dependent anharmonic oscillator equation

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Abstract. A one-dimensional time dependent anharmonic oscillator is considered in the frame of transformation group techniques. Asymptotic solutions are obtained for the class of time decreasing coefficients of the equation.

1. Introduction

The 'time varying frequency harmonic oscillator equation' $d^2q/dt^2+\omega^2(t)q=0$ is one of the favourite topics of mathematical physics. It allows one to study interesting concepts such as the adiabatic invariant one, for example. Recently a new treatment was introduced (Burgan *et al* 1978a) (Leach 1977) consisting in a 'rescaling' on q and in the introduction of a new time. These transformations, which form a group, are characterised by a function $C(t)$ which is chosen in order to simplify the problem. Especially when $\omega^2(t)\to 0$ as $t\to\infty$, we see how $C(t)$ must be selected to obtain the asymptotic form of the solution and so know more precisely if the oscillator always feels the force field (describing a spiral in phase space) or becomes free of it (asymptotically moving as a free particle, a behaviour connected to the breaking of the adiabatic invariants). It has been shown that the limiting case was for $\omega(t)$ varying as t^{-1}. For a slower decay, the WKB approximation is correct for all values of t. For a faster decay, the particle becomes asymptotically free. This is detailed in Burgan *et al* (1978b).

The purpose of this paper is to extend these results to the following equation

$$d^2q/dt^2+A\omega^2(t)q+B\lambda(t)q^3=0\ldots+A\omega^2(t)q+\ldots \tag{1}$$

with $A\in\mathbb{R}$ and $B\in\mathbb{R}$.

Preliminary results on this equation have been reported by Burgan *et al* (1980) and a study of an equation with a q^2 (instead of q^3 term) is given by Leach (1979).

We suppose that for a large enough time we have the relations

$$\omega^2(t)=(1+\Omega t)^{-\mu}$$
$$\lambda(t)=(1+\Omega t)^{-\nu}$$

with $\mu\in\mathbb{R}^+$, $\nu\in\mathbb{R}^+$.

For commodity, we shall take them to be true on the whole interval $t\in[0,\infty[$. We look for the asymptotic solutions using the group transformation technique under the three following rules (Burgan 1978a).

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(i) We use the generalised canonical transformation characterised by the arbitrary function $C(t)$ transforming the x, $v = \mathrm{d}x/\mathrm{d}t$, t space into the new space ξ, $\eta = \mathrm{d}\xi/\mathrm{d}\theta$, θ with conservation of the Hamiltonian formalism (see below).

(ii) We try by a proper selection of the arbitrary function $C(t)$ to transform the infinite extension of t into a finite interval (time renormalisation).

(iii) This last objective must not be obtained at the expense of coefficients of the new equation going to infinity. The finiteness of these coefficients has priority on (ii) and sometimes we simply obtain 'time compression' instead of time renormalisation.

The generalised canonical transformation is characterised by a function $C(t)$ such that the variation of q is expressed both through a rescaling characterised by $C(t)$ and a variation of ξ with the new time θ.

$$q = \xi(\theta)C(t) \qquad \theta = \theta(t). \tag{2}$$

Introducing (2) into (1) we obtain

$$C\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\frac{\mathrm{d}^2\xi}{\mathrm{d}\theta^2}+\left\{\frac{d^2C}{\mathrm{d}t^2}+A\omega^2(t)C\right\}\xi+\frac{\mathrm{d}\xi}{\mathrm{d}\theta}\left\{2\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}C}{\mathrm{d}t}+C\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right\}+B\lambda C^3\xi^3=0.$$

We want to keep a Hamiltonian formalism in the new space which consequently implies that the term $\mathrm{d}\xi/\mathrm{d}\theta$ cancels. Consequently

$$\mathrm{d}\theta/\mathrm{d}t = 1/C^2 \tag{3}$$

and (1) becomes

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}\theta^2}+\left[A\omega^2C^4+C^3\frac{d^2C}{\mathrm{d}t^2}\right]\xi+B\lambda C^6\xi^3=0. \tag{4}$$

In (4) we will call $A\omega^2C^4\xi$, $\xi C^3\mathrm{d}^2C/\mathrm{d}t^2$ and $B\lambda C^6\xi^3$, the rescaled linear physical field, the transformation field and the rescaled non-linear physical field, respectively, a quite natural designation for the three last terms of (4).

Since we have supposed that ω^2 and λ vary as $(1+\Omega t)^{-\mu}$ and $(1+\Omega t)^{-\nu}$, we take $C(t)$ varying as $(1+\Omega t)^{\gamma}$; γ is left to our choice with the rules already indicated. No infinity must appear in the ξ and ξ^3 coefficients, and within this constraint γ must be as big as possible in order to renormalise the time or at least to compress it as much as we can.

We shall distinguish two cases in this study: $\nu < \frac{3}{2}\mu$ and $\nu > \frac{3}{2}\mu$. This inequality comes from the fact that we want to keep finite both $A\omega^2C^4$ and $B\lambda C^6$; the first term implies that $\gamma \leqslant \frac{1}{4}\mu$ and the second that $\gamma \leqslant \frac{1}{6}\nu$. Now if $\gamma < \frac{3}{2}\mu$, taking $\gamma \leqslant \frac{1}{6}\nu$ implies that $\gamma < \frac{1}{4}\mu$ and *vice versa*, if $\nu > \frac{3}{2}\mu$, taking $\gamma \leqslant \frac{1}{4}\mu$ implies that $\gamma < \frac{1}{6}\nu$.

2. Quickly decreasing linear term

We first consider $\nu < \frac{3}{2}\mu$.

The choice of $C(t)$ is shown in table 1. This choice is easy to understand. We will comment on it, starting from the last line.

(a) $\nu > 6$.

The time is renormalised ($\Omega\theta$ varies from 0 to 1), the coefficients of ξ and ξ^3 remain finite in the interval $0 < \theta < \Omega^{-1}$ and take the value 0 for $\Omega\theta = 1$; the solution of (4) gives the limiting point ξ_l and consequently $\forall A$ and B, the motion in q space corresponds to a free particle one.

$$q_{\text{as}} \propto (1+\Omega t)$$

(where the subscript as stands for asymptotic).

(b) $3<\nu<6$.

The time undergoes a logarithmic compression. Taking $C(t)=(1+\Omega t)^{\nu/6}$ would give a time renormalisation, and would of course simplify the term λC^6, but this is impossible because of (iii) since the term $C^3\ddot{C}$ would blow up. We consequently take $\gamma=\frac{1}{2}$ which keeps the transformation field finite. Notice that the only other possibility for γ is to take the value one, but ν must be bigger than or equal to 6 to keep finite the nonlinear rescaled field.

Table 1.

ν	$C(t)$	$C^3\ddot{C}$	ω^2C^4	λC^6	$\Omega\theta$
$0<\nu<3$	$(1+\Omega t)^{\nu/6}$	$\Omega^2\frac{1}{6}\nu(\frac{1}{6}\nu-1)$ $(1+\Omega t)^{\frac{2}{3}\nu-2}$	$A(1+\Omega t)^{\frac{2}{3}\nu-m}$	B	$[1/(1-\frac{1}{3}\nu)]$ $\times[(1+\Omega t)^{1-\frac{1}{3}\nu}-1]$
$\nu=3$ implying $\mu>2$	$(1+\Omega t)^{1/2}$	$-\frac{1}{4}\Omega^2$	$A\exp[\Omega\theta(2-\mu)]$	B	$\ln(1+\Omega t)$
$3<\nu<6$ implying $\mu>2$	$(1+\Omega t)^{1/2}$	$-\frac{1}{4}\Omega^2$	$A\exp[\Omega\theta(2-\mu)]$	$B\exp[\Omega\theta(3-\nu)]$	$\ln(1+\Omega t)$
$\nu=6$ implying $\mu>4$	$1+\Omega t$	0	$A(1-\Omega\theta)^{-(4-\mu)}$	B	$\Omega t/(1+\Omega t)$
$\nu>6$ implying $\mu>4$	$1+\Omega t$	0	$A(1-\Omega\theta)^{-(4-\mu)}$	$B(1-\Omega\theta)^{-(6-\nu)}$	$\Omega t/(1+\Omega t)$

The equation becomes

$$d^2\xi/d\theta^2+[A\exp-\Omega\theta(\mu-2)-\tfrac{1}{4}\Omega^2]\xi+[B\exp-\Omega\theta(\nu-3)]\xi^3=0. \tag{5}$$

When $\theta\to\infty$, then the term $A\exp[-\Omega\theta(\mu-2)]$ is negligible compared with $(-\frac{1}{4}\Omega^2)$. One has to evaluate the respective importance of the terms in ξ and ξ^3. We notice that the coefficient of the ξ^3 term goes to zero. Consequently, we first suppose that (5) is equivalent to

$$d^2\xi/d\theta^2-\tfrac{1}{4}\Omega^2\xi=0. \tag{6}$$

The solution of (6) is proportional to $\exp(\Omega\theta/2)$ and we look at the ratio R of the ξ and ξ^3 terms

$$R=\left|\frac{B[\exp\Omega\theta(3-\nu)]\xi^3}{\frac{1}{4}\Omega^2\xi}\right| \text{ behaves as } \exp\Omega\theta(4-\nu).$$

Consequently

(b′) If $6>\nu>4$, $R\to 0$ when $\theta\to\infty$ and (5) is equivalent to (6); we have, asymptotically, $\xi(\theta)\propto\exp(\Omega\theta/2)$ and $C(t)=(1+\Omega t)^{1/2}=\exp(\Omega\theta/2)$. Consequently $q=$

$\xi(\theta)C(t)$ behaves as $\exp(\Omega\theta)$ i.e. as $(1+\Omega t)$ and we have, for large time, the motion of a free particle.

(b″) If $4>\nu>3$, the following transformation $\xi=\psi\exp[(\nu-3)\Omega\theta/2]$ leads to the equation

$$d^2\psi/d\theta^2+\Omega(\nu-3)d\psi/d\theta+[(\nu-4)(\nu-2)\Omega^2/4+A\exp\Omega\theta(2-\mu)]\psi+B\psi^3=0. \qquad (7)$$

Since $\mu>2$, the term $A\exp\Omega\theta(2-\mu)\to 0$ (independently of the sign of A), and if $B>0$, (7) shows that we have a damped motion $(\nu-3>0)$ in a potential of the form given in figure 1. Consequently the solution will reach one of the holes given by $\pm\psi_l$ with

$$\psi_l=[4B/(4-\nu)(\nu-2)\Omega^2]^{1/2}$$

the sign will be fixed by the initial conditions and we have two limiting points in the ξ, $d\xi/d\theta$ phase plane.

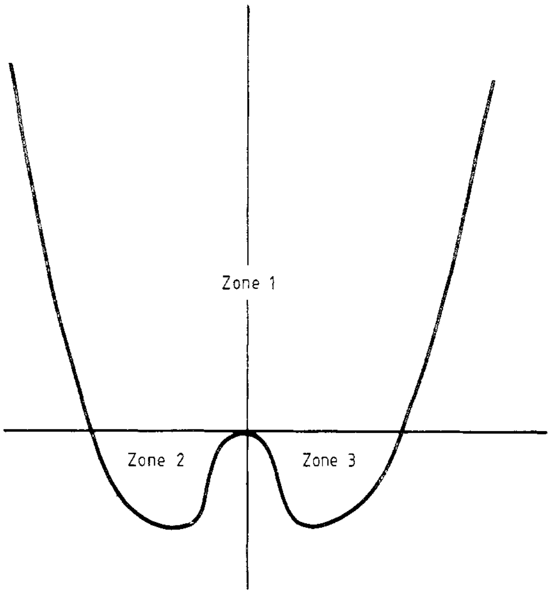

Figure 1. Asymptotic potential for $4>\nu>3$, $B>0$.

So the asymptotic solution, after the damping of the initial oscillation is

$$q_{as}=\pm[4B/(4-\nu)(\nu-2)\Omega^2]^{1/2}(1+\Omega t)^{\frac{1}{2}\nu-1}. \qquad (8)$$

We must notice that (8) is exactly the self-similar solution obtained by solving (1) where we have first neglected the linear term (which unless $\mu=2$ blocks any possible self-similar solution). This self-similar solution obtained by writing $q=K(1+\Omega t)^\alpha$ in the simplified equation and properly identifying α and K, consequently plays the role of an attractor.

If $B<0$, ψ goes to infinity very quickly, and more precisely, reaches an infinite value in a finite time θ, indicating an explosive instability.

(c) If $\nu=3$, equation (5) becomes

$$d^2\xi/d\theta^2+[A\exp-\Omega\theta(\mu-2)-\tfrac{1}{4}\Omega^2]\xi+B\xi^3=0. \qquad (9)$$

When $\theta \to \infty$, then $A \exp[-\Omega\theta(\mu-2)] \to 0$ since $\mu > \frac{2}{3}\nu = 2$. $\mathrm{d}^2\xi/\mathrm{d}\theta^2 = -\mathrm{d}V/\mathrm{d}\xi = -B\xi^3 + \frac{1}{4}\Omega^2\xi$, the aspect of the potential V is the same as before when $B > 0$. Since there is no damping, the ξ solution is oscillatory, in zone I, II or III according to the initial conditions. We also remark that the case $\mu = 2$, $\nu = 3$ can be completely integrated with the use of elliptic integrals.

When $B < 0$, we again have an explosive instability and ξ goes to infinity in a finite time. One can compute the motion by integrating once the equation

$$\mathrm{d}^2\xi/\mathrm{d}\theta^2 - \tfrac{1}{4}\Omega^2\xi - |B|\xi^3 = 0 \tag{10}$$

we obtain

$$\mathrm{d}\xi/\mathrm{d}\theta = \pm(\tfrac{1}{4}\Omega^2\xi^2 + \tfrac{1}{2}|B|\xi^4 + C)^{1/2} \tag{11}$$

the constant is determined by the initial conditions. Excluding the particular condition $\xi = 0$, $\mathrm{d}\xi/\mathrm{d}\theta = 0$ when $\theta = 0$, we have to consider one of the integrals

$$\int_{\xi_0}^{\infty} \text{ or } \int_{-\infty}^{\xi_0} \mathrm{d}\xi(\tfrac{1}{4}\Omega^2\xi^2 + \tfrac{1}{2}|B|\xi^4 + C)^{-1/2}.$$

These integrals are bounded and the integration on ξ gives a finite time. The diagrams 2(*a*) and 2(*b*) show the evolution of ξ with time.

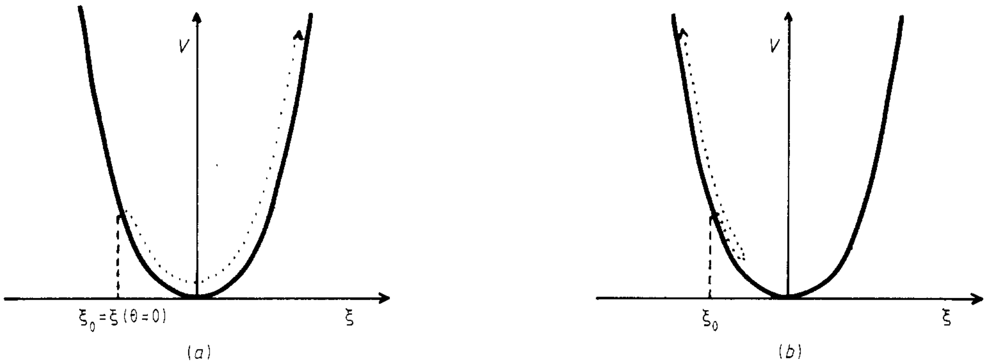

Figure 2. Time evolution of ξ according to the two possible classes of initial conditions ($\nu < \frac{3}{2}\mu$, $\nu = 3$, $B < 0$): (*a*) $\mathrm{d}\xi/\mathrm{d}\theta > C$; (*b*) $\mathrm{d}\xi/\mathrm{d}\theta < C$. $C \approx (\frac{1}{4}\Omega^2\xi_0^2 + \frac{1}{2}B\xi_0)^{1/2}$.

(d) If $\nu < 3$, we take $\gamma = \nu/6$ and terms in $C^3\mathrm{d}^2C/\mathrm{d}t^2$ and ω^2C^4 decreases to zero. The asymptotic equation is

$$\mathrm{d}^2\xi/\mathrm{d}\theta^2 + B\xi^3 = 0 \tag{12}$$

which solution is a nonlinear oscillatory movement if $B > 0$ and an explosive instability if $B < 0$ (Davis 1962).

3. Quickly decreasing non-linear terms

Second case: $\nu > \frac{3}{2}\mu$.

The choice of $C(t)$ is shown in table 2. The analysis of the different cases is performed in the same way as before.

Table 2.

μ	ν	$C(t)$	$C^3\ddot{C}$	ω^2C^4	λC^6	$\Omega\theta$
$0<\mu<2$		$(1+\Omega t)^{\mu/4}$	$\frac{1}{4}\mu(\frac{1}{4}\mu-1)$ $(1+\Omega t)^{\mu-2}$	A	$B(1+\Omega t)^{\frac{3}{2}\mu-\nu}$	$[1/(1-\frac{1}{2}\mu)]$ $\times[(1+\Omega t)^{1-\frac{1}{2}\mu}-1]$
$\mu=2$	$\nu>3$	$(1+\Omega t)^{1/2}$	$-\frac{1}{4}\Omega^2$	A	$B\exp[\Omega\theta(3-\nu)]$	$\ln(1+\Omega t)$
$2<\mu<4$	$\nu>3$	$(1+\Omega t)^{1/2}$	$-\frac{1}{4}\Omega^2$	$A\exp[\Omega\theta(2-\mu)]$	$B\exp[\Omega\theta(3-\nu)]$	$\ln(1+\Omega t)$
$\mu=4$	$\nu>6$	$1+\Omega t$	0	A	$B(1-\Omega\theta)^{\nu-6}$	$\Omega t/(1+\Omega t)$
$\mu>4$	$\nu>6$	$1+\Omega t$	0	$A(1-\Omega\theta)^{\mu-4}$	$B(1-\Omega\theta)^{\nu-6}$	$\Omega t/(1+\Omega t)$

(a) $\mu>4$.
We can take $\gamma=1$ and for all values of A and B, the asymptotic motion is the one of a free particle

$$q_{\mathrm{as}}\propto(1+\Omega t).$$

(b) The case $2<\mu<4$ is divided in two parts according to the values of ν since we must take $\gamma=\frac{1}{2}$ and we recover equation (5). The analysis proceeds as before and for $\nu>4$ we get a free particle motion, while for $4>\nu>3$ we get the solution given by (8).

(c) $\mu=2$.
The equation is

$$\mathrm{d}^2\xi/\mathrm{d}\theta^2+(A-\tfrac{1}{4}\Omega^2)\xi+B\exp[\Omega\theta(3-\nu)]\xi^3=0. \tag{13}$$

(C_1) If $(A-\frac{1}{4}\Omega^2)>0$, since asymptotically the coefficient of the term in ξ^3 goes to zero and the solution ξ is oscillatory, the asymptotic amplitude goes as $t^{1/2}$.

(C_2) If $(A-\frac{1}{4}\Omega^2)<0$, (13) must be studied as (5); the solution $\xi=\exp(-A+\frac{1}{4}\Omega^2)^{1/2}\theta$ must be introduced in (13) and the ratio of the last two terms of the LHS of (13) must be computed. Two cases appear:

($C_{2'}$) $\nu>3+2(\frac{1}{4}-A/\Omega^2)^{1/2}$.
Asymptotically, the term in ξ^3 is negligible, then the asymptotic solution in $\exp\Omega\theta(\frac{1}{4}-A/\Omega^2)^{1/2}$ and

$$q\propto(1+\Omega t)^{\frac{1}{2}+(\frac{1}{4}-A/\Omega^2)^{1/2}}$$

($C_{2''}$) $\nu<3+2(\frac{1}{4}-A/\Omega^2)^{1/2}$.
With the following transformation $\xi=\psi\exp[\Omega\theta(\nu-3)/2]$, (13) becomes

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\theta^2}+(\nu-3)\Omega\frac{\mathrm{d}\psi}{\mathrm{d}\theta}+\left[\left(\frac{\nu-3}{2}\Omega\right)^2-\left(\frac{\Omega^2}{4}-A\right)\right]\psi+B\psi^3=0. \tag{14}$$

For $B>0$ (14) describes a damped motion ($\nu>3$) in a potential which has the same form as given in figure 1(a). ψ goes to

$$\psi_l=\pm\left\{B\Big/\left[\left(\frac{\Omega^2}{4}-A\right)-\left(\frac{\nu-3}{2}\right)^2\Omega^2\right]\right\}^{1/2}$$

and

$$q_{\mathrm{as}}=\pm\left\{B\Big/\left[\left(\frac{\Omega^2}{4}-A\right)-\left(\frac{\nu-3}{2}\right)^2\Omega^2\right]\right\}^{1/2}(1+\Omega t)^{\nu/2-1}. \tag{15}$$

We notice that in this case ($\mu = 2$) equation (1), this time without having to neglect any term, has a self-similar solution in $t^{(\nu/2-1)}$. This solution is exactly (15).

For $B < 0$ ψ goes to infinity in a finite time (explosive instability).

(C_3) If $A = \frac{1}{4}\Omega^2$, the term in ψ disappears in (14). Since $\nu > 3$ the last term in the RHS of (13) goes to zero and we obtain asymptotically the free motion solution in ξ, θ space, i.e. $\xi \propto \theta$.

(d) $0 < \mu < 2$, the coefficient of the transformation field and of the non-linear rescaled field go to zero; the asymptotic equation is

$$d^2\xi/d\theta^2 + A\xi = 0. \tag{16}$$

If $A > 0$, the solution $\xi(\theta)$ stays finite and it is correct to neglect the other terms, especially the one in ξ^3, since its coefficient goes to zero with θ.

If $A < 0$, the solution obtained by neglecting the ξ^3 term goes to infinity and consequently, we must check the consistency of the hypothesis. We find in that case that the ξ^3 term always has a role to play. Consequently we introduce the following transformation

$$\xi = \psi(1+\Omega t)^{-d/2} = \psi(1+c\Omega\theta)^{-d/2}$$

where

$$c = 1 - \tfrac{1}{2}\mu \qquad (c > 0)$$
$$d = (3\mu - 2\nu)/(2-\mu) \qquad (d < 0) \tag{17}$$

applied to the complete equation (4) with the choice of $C(t)$ as indicated in the first line of table 2 gives

$$\frac{d^2\psi}{d\theta^2} - d\Omega c(1+c\Omega\theta)^{-1}\frac{d\psi}{d\theta} + \ldots$$
$$\left[\frac{d}{2}\left(\frac{d}{2}+1\right)\frac{c^2\Omega^2}{(1+c\Omega\theta)^2} + A + \frac{\mu}{4}\left(\frac{\mu}{4}-1\right)\frac{\Omega^2}{(1+c\Omega\theta)^2}\right]\psi + B\psi^3 = 0. \tag{18}$$

When $\theta \to \infty$, (18) becomes

$$\frac{d^2\psi}{d\theta^2} + \frac{|d|\Omega c}{1+c\Omega\theta}\frac{d\psi}{d\theta} + A\psi + B\psi^3 = 0 \tag{19}$$

which represents the damped motion of a particle in a potential $\frac{1}{2}A\psi^2 + \frac{1}{4}B\psi^4$. If $B > 0$, we recover the two holes potential of figure 1 and the solution goes to $\psi_l = \pm(|A|/B)^{1/2}(A < 0)$. Taking into account $q = \xi C(t)$ and $\xi = \psi(1+c\Omega\theta)^{-d/2} = \psi(1+\Omega t)^{-cd/2}$ we obtain

$$q_{as} = \pm(|A|/B)^{1/2}(1+\Omega t)^{(\nu-\mu)/2}. \tag{20}$$

It is worth noticing that (20) is the solution obtained by neglecting the inertial term in (1) and observing that the repulsive (since $A < 0$) linear force is just balanced by the attractive non-linear one.

If $B < 0$, ψ and consequently q goes to infinity in a finite time as already seen before.

To summarise the results obtained, one can present figure 3 and table 3.

The figure represents the parameter space μ, ν and is divided into eight zones labelled respectively from (1) to (8) and delineated by the oblique line $\nu = \frac{3}{2}\mu$, three horizontal lines $\nu = 3$, $\nu = 4$, and $\nu = 6$, and two vertical lines $\mu = 2$ and $\mu = 4$.

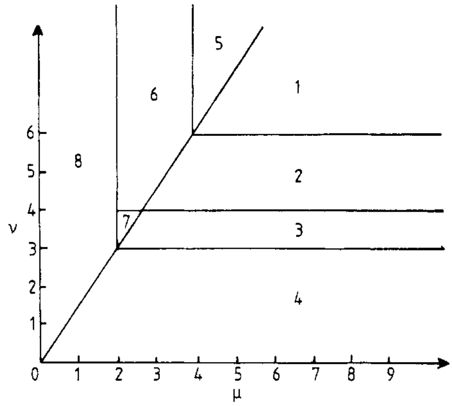

Figure 3. Summary of the results.

Table 3. Summary of the results

Superzone	$A>0$ $B>0$	$A>0$ $B<0$	$A<0$ $B>0$	$A<0$ $B<0$
I = (1) + (2) + (5) + (6)		Free particle		
II = (3) + (7)	$(1+\Omega t)^{(\frac{1}{2}\nu-1)}$	expl	$(1+\Omega t)^{(\frac{1}{2}\nu-1)}$	expl
III = (4)	NL oscill $*(1+\Omega t)^{\nu/6}$	expl	NL oscill $*(1+\Omega t)^{(\nu/6)}$	expl
IV = (8)	L oscill $*(1+\Omega t)^{\mu/4}$	L oscill $*(1+\Omega t)^{\mu/4}$	$(1+\Omega t)^{(\nu-\mu)/2}$	expl

For each zone we have the asymptotic solution, but some of the zones can be grouped together (forming four superzones) where the physical behaviour of the asymptotic solutions is the same—although we may have used different values of γ in the different zones of the same superzone. For example, in (1) and (5) we have been able to renormalise the time with the consequence of a free-particle asymptotic motion while in (2) and (6) we have simply taken $\gamma=\frac{1}{2}$ and checked that the solution $\exp \Omega\theta/2$ obtained by neglecting the ξ^3 term (the coefficient of which vanishes) does indeed give a negligible term when introduced into the full equation. The superzones are formed by the union of the zones indicated in the first column. The asymptotic behaviour is given with L standing for linear oscillation, NL standing for non-linear oscillation and expl standing for explosive instability. Moreover, the time behaviour of the amplitudes for the linear and non-linear oscillation is indicated by *.

4. Conclusion

The generalised canonical transformation preserving the Hamiltonian formalism has been applied to problems where the forces include a linear and non-linear term, both decreasing asymptotically with some power of the time. One scale, for example, the length scale, characterised by $C(t)$ is at our disposal and all the others are subsequently derived from $C(t)$. By a proper selection of $C(t)$ we can obtain in the new time phase space, an equation which can be asymptotically solved giving different limit cycles on limit points. Moreover, an interesting result is that the border of the regions in the parameter space $\mu\nu$ where the solutions take different forms (bifurcation lines) are partly given by the possibility of finding self similar solutions. Moreover these solutions appear here as an attractor in superzone II. This frontier status of the self-similar solution is, *a posteriori*, not very surprising since such solutions are obtained by forcing a balance between the different terms, which is time preserved. The self-similar solutions are consequently always interesting to obtain, a result which has been found in gravitational fluid mechanics, quantum adiabatic invariants and stellar dynamics.

References

Burgan J R, Gutierrez J, Munier A, Fijalkow E and Feix M R 1978a *Strongly coupled plasma*, ed G Kalman (New York: Plenum) p 597

Burgan J R, Feix M R, Fijalkow E, Gutierrez J and Munier A 1978b *Applied Inverse Problem*, ed P C Sabatier, *Lecture Notes in Physics* **85** (Berlin: Springer) pp 248–73

Burgan J R, Feix M R, Fijalkow E, Munier A and Nakach R 1980 *J. Physique* **41** C3 377–82

Davis H T 1962 *Introduction to Nonlinear Differential and Integral Equations* (New York: Dover)

Leach P G L 1977 *J. Math. Phys.* **18** 1608–11

—— 1979 *J. Math. Phys.* **20** 96–100